The Jeans Criterion and the Collapse of Clouds

Stilianos Louca

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Abstract

This article introduces the Jeans stability criterion for interstellar clouds from a non-relativistic point of view. Three different approaches to deriving similar forms of the criterion are presented.

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1 Introduction

Molecular clouds (MC) are interstellar clouds, consisting mainly of molecules (e.g. H_2). Their typical temperatures vary in the order of 10 K, their densities in the order of 10^{-19} Kg $\cdot$ m^{-3}.

Molecular clouds are believed to be an important source of star formations, which take place as they collapse under their internal gravitational force. These collapses can be triggered by gravitational instabilities and perturbations, shock waves from supernova explosions or even strong radiation pressure of nearby stars. Their typical lifetime is in the order of their free-fall time-scale, that is, just a few million years, and it is believed that once molecular clouds are formed, they lead to star formations quite quickly.

Of particular interest are *Giant molecular clouds* (GMC), which can reach diameters of up to 10^2 parsecs and masses up to $10^6 M_\odot$. As a consequence, they are birthplace of big star clusters and current star formations can often be observed within them. A well known GMC is the Orion Molecular Cloud.

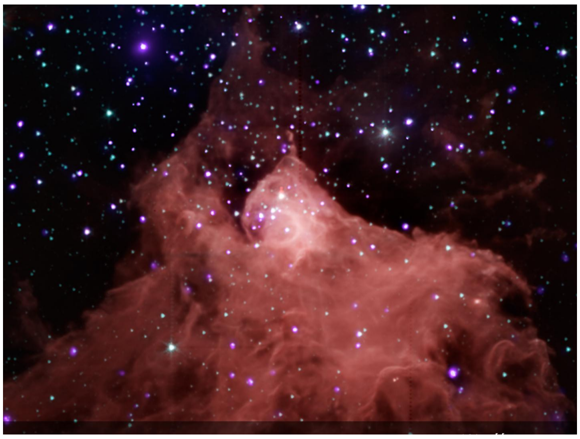

Figure 1.1: Composite image of Cepheus B, a GMC about 2400 Lj away. Red, green & blue data is in infrared, violet in X-Ray spectrum. The image span is about 3 pc.[6]

In the following, we shall describe a simple, well known criterion for the stability of clouds, from a non-relativistic point of view. This so called Jeans criterion was first derived by Sir James Jeans, who showed that a cloud of given density could, at sufficiently low temperature and sufficiently big diameter, collapse under its internal gravitational pressure.

The material for the following article, was taken mainly from Kippenhahn & Weigert[1], Phillips[2] and Longair[4].

2 The Jeans Criterion

Dating back to the work of Jeans, the problem of growth of small gravitational perturbations is of central importance in understanding stability of MCs. As it turns out, wether a MC collapses into a star or not, highly depends on its dimensions, density ρ and temperature T. Two of the key concepts arising in the treaty of these conditions, are the so called *Jeans length* and *Jeans mass*.

Non-relativistic investigations result in a Jeans length proportional to $c_s \cdot (G\rho)^{-\frac{1}{2}}$, where c_s is the (isothermal) speed of sound, which describes the maximum perturbation length (or GMC diameter), for which perturbations remain oscillations, or for which, mass aggregations are stable for that matter.

There are quite a few different models one may use to obtain this characteristic length, each providing a different insight into the subject. In the following, we shall illustrate 3 of them.

2.1 The Jeans stability criterion for gravitational perturbations

We shall consider small gravitational perturbations on the background of an inviscid fluid with density distribution ρ, velocity distribution $\mathbf{v}$, temperature distribution T and pressure distribution p. Φ shall denote the gravitational potential and shall, together with p, be the sole source of forces on the fluid.

Beginning with the Euler equations of motion

$$\frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v}) \cdot \mathbf{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi \quad , \tag{2.1}$$

the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad , \tag{2.2}$$

Newton's law

$$\Delta\Phi = 4\pi G\rho \tag{2.3}$$

and the equation of state

$$p = p(\rho, T) \tag{2.4}$$

we assume:

(i) $\rho, p, \mathbf{v}, \Phi$ to be static solutions of eq. (2.1), (2.2), (2.3) and (2.4).

(ii) T, ρ to be constant (over space).

(iii) W.l.o.g. $\mathbf{v} = 0$.

Now, for small, isothermal perturbations $\delta\mathbf{v},\ \delta\rho,\ \delta p,\ \delta\Phi\ \sim \mathcal{O}(\varepsilon)$ (ε: scaling parameter) these equations lead to

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{1}{\rho}\nabla\delta p + \frac{\delta\rho}{\rho^2}\nabla p - \nabla\delta\Phi + \mathcal{O}(\varepsilon^2)$$

$$\frac{\partial \delta\rho}{\partial t} + \rho \cdot (\nabla \cdot \delta\mathbf{v}) = \mathcal{O}(\varepsilon^2) \tag{2.5}$$

and

$$\Delta\delta\Phi = 4\pi G\delta\rho \quad . \tag{2.6}$$

Using $\nabla p = c_s^2 \nabla\rho$ and $\delta p = c_s^2\delta\rho$, with $c_s^2 := \left(\frac{\partial p}{\partial \rho}\right)_{T:\text{const}}$ as the isothermal speed of sound, we obtain

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\nabla\left(c_s^2\frac{\delta\rho}{\rho} + \delta\Phi\right) + \mathcal{O}(\varepsilon^2) \tag{2.7}$$

Taking the divergence of (2.7) and using eq. (2.5) & (2.6), we get

$$-\frac{\partial^2}{\partial t^2}\frac{\delta\rho}{\rho} \overset{(2.6)}{=} \frac{\partial}{\partial t}(\nabla\cdot\delta\mathbf{v}) + \mathcal{O}(\varepsilon^2) \overset{(2.6)}{=} -c_s^2\Delta\left(\frac{\delta\rho}{\rho}\right) - 4\pi G\delta\rho + \mathcal{O}(\varepsilon^2), \tag{2.8}$$

or equivalently

$$\frac{\partial^2\delta\rho}{\partial t^2} = c_s^2\Delta\delta\rho + 4\pi G\rho\cdot\delta\rho + \mathcal{O}(\varepsilon^2). \tag{2.9}$$

Ignoring the 2nd order term $\mathcal{O}(\varepsilon^2)$, we make the ansatz $\delta\rho \sim \exp\left[i(\mathbf{kx} - \omega t)\right]$ of a plane wave and obtain

$$\boxed{\omega^2 = c_s^2\mathbf{k}^2 - 4\pi G\rho} \tag{2.10}$$

as necessary & sufficient condition for solvability. Eq. (2.10) is the dispersion relation for plane-wave perturbations. Depending on the background density and wavelength $\lambda = 2\pi/\|\mathbf{k}\|$, ω may become real or imaginary, corresponding to oscillating or *exploding/imploding* solutions of (2.9) respectively. The wavelength limit

$$\boxed{\lambda_J = c_s \cdot \sqrt{\frac{\pi}{G\rho}}} \tag{2.11}$$

is called the *Jeans length*, and represents an upper limit for the scale of non-exploding/imploding density perturbations. Assuming an ideal gas of particle mass μ and the equation of state $p = \rho kT/\mu$, the (isothermal) speed of sound is given by $c_s = \left(\frac{\partial p}{\partial \rho}\right)_T = \sqrt{kT/\mu}$. In that case, eq. (2.11) becomes

$$\boxed{\lambda_J^{\text{ideal}} = \sqrt{\frac{\pi kT}{\mu G\rho}}.} \tag{2.12}$$

In the above calculations, we assumed the temperature T to be constant even under the emerging perturbation. This of course requires the cloud to be adequately transparent, for radiation to provide for thermal equilibrium.

Note: The assumption of $\mathbf{v}, T, \rho$ (and therefore p, Φ) being constant (homogeneity, isotropy) inevitably leads to $\rho = 0$, which at first sight represents a flaw in the above derivation[1]. A mathematically more rigorous approach can be found in [3].

2.2 The Virial theorem derivation of the Jeans length

Consider a mass distribution (*cloud*) of particles with no internal degrees of freedom, density distribution ρ and Temperature distribution T. From the virial theorem, the time-average kinetic energy $\langle E_{\text{kin}}\rangle$ of the system is given by

$$\langle E_{\text{kin}}\rangle = -\left\langle \frac{1}{2}\int \mathbf{f}(\mathbf{r})\cdot\mathbf{r}\; d^3\mathbf{r}\right\rangle, \tag{2.13}$$

where $\mathbf{f}(\mathbf{r})$ is the force-density at position $\mathbf{r}$. Assuming that all inner forces of the cloud are of gravitational nature, we obtain

$$\langle E_{\text{kin}}\rangle = -\frac{1}{2}\langle E_{\text{pot}}\rangle, \tag{2.14}$$

with $\langle E_{\text{pot}}\rangle$ being its average total potential energy. Now assume:

(i) The cloud to be in static equilibrium.

(ii) The cloud to be spherical with radius R.

(iii) ρ, T to be constant within its radius.

Then the average kinetic energy of each particle is given by $\frac{3}{2}kT$ and $\langle E_{\text{kin}}\rangle = E_{\text{kin}}$, $\langle E_{\text{pot}}\rangle = E_{\text{pot}}$. In particular

$$E_{\text{kin}} = \frac{4\pi R^3\rho}{3}\cdot\frac{3kT}{2\mu} = 2\pi\frac{\rho}{\mu}kTR^3, \tag{2.15}$$

where μ is the particle mass. Similarly,

$$E_{\text{pot}} = -\int \frac{4\pi r^3\rho}{3}\cdot\frac{\rho G}{r}\cdot 4\pi r^2\; dr = -\frac{16\pi^2}{15}G\rho^2R^5 \quad . \tag{2.16}$$

[1] This fact came to be known as the so called *Jeans swindle*.

Together with the virial theorem (2.14), eq. (2.15) and (2.16) yield the so called *Jeans radius*

$$\boxed{R = R_J = \sqrt{\frac{15kT}{4\pi\mu G\rho}},} \tag{2.17}$$

which represents, for a given density ρ and temperature T, an upper limit for the radius of stable, spherical, homogeneous mass aggregations. Up to a numerical factor of order 1, it coincides with the Jeans length (2.12) obtained through linear perturbation theory.

2.3 Hydrodynamic derivation of the Jeans length

We consider a static, spherical mass aggregation of density distribution ρ, Temperature distribution T and pressure distribution p. We assume that the only forces acting on the particles are gas-presure and gravitational. The former results in a local, radial force density $f_{\mathrm{p}}(r) = -\frac{dp}{dr}$, the later in the force density $f_{\mathrm{g}}(r) = -m(r)G\rho/r^2$, whereas

$$m(r) := \int_{r' \leq r} \rho(\mathbf{r})\ d^3\mathbf{r} \tag{2.18}$$

is the total mass within radius $\leq r$.

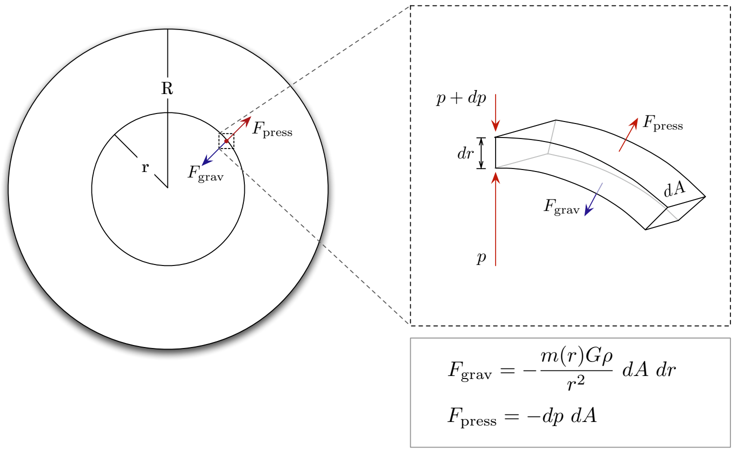

Figure 2.1: Forces acting on a mass-element at central distance r. Gravitational forces only result from mass located within r. Pressure forces actually result from a pressure gradient!

Equilibrium of the system implies

$$-\frac{dp}{dr} = f_{\mathrm{p}} = -f_{\mathrm{g}} = \frac{Gm(r)\rho(r)}{r^2}. \tag{2.19}$$

Multiplying eq. (2.19) with $4\pi r^3$ and integrating both sides, results in

$$E_{\text{pot}} = -\int \frac{Gm(r)}{r}\rho(r)\ d^3\mathbf{r} = -\int 4\pi G m(r)\rho(r) r\ dr \overset{(2.19)}{=} \int 4\pi r^3 \frac{dp}{dr}\ dr$$

$$= \underbrace{4\pi r^3 p\Big|_0^\infty}_{0} - 3\underbrace{\int p(r)\cdot 4\pi r^2\ dr}_{\langle p\rangle\cdot V} = -3\langle p\rangle \cdot V, \tag{2.20}$$

with $\langle p \rangle$ as the space-average pressure over the system volume V. For an ideal gas of non-relativistic particles $p = \rho kT/\mu$, so that $\langle p \rangle = \langle \rho \rangle kT/\mu$, and under the assumption of constant temperature T, eq. (2.20) becomes

$$E_{\text{pot}} = -\frac{3kT}{\mu} M, \tag{2.21}$$

with M as the total mass of the system. Furthermore, eq. (2.16) implies

$$E_{\text{pot}} = -f \cdot \frac{GM^2}{R}, \tag{2.22}$$

with f being a numerical factor of order 1, depending on the exact density distribution ρ ($f = 3/5$ for constant density). Consequently, (2.21) results in a radius

$$\boxed{R = \sqrt{\frac{9kT}{f 4\pi\mu G \langle \rho \rangle}},} \tag{2.23}$$

which is in exact accordance with the results of 2.2, provided ρ = const and thus $f = 3/5$. But this should come to no surprise, as the assumptions made in both models were essentially those of an ideal gas, of constant temperature and density, at equilibrium within a radius R.[2]

The corresponding limit mass of the aggregation turns out to be

$$\boxed{M_J = \left(\frac{5kT}{\mu G}\right)^{\frac{3}{2}} \cdot \left(\frac{3}{4\pi\rho}\right)^{\frac{1}{2}}.} \tag{2.24}$$

It is the maximum mass, at given density ρ and temperature T, a spherical cloud may have without collapsing.

3 Collapse and fragmentation

3.1 The process of collapse

When a cloud of given density ρ and temperature T is large enough, gravitational forces exceed internal pressure and cause the cloud to collapse. As long as the density ρ of the collapsing cloud remains adequately low for the cloud to be transparent, the released thermal energy is radiated into the universe and the temperature remains approximately constant. As eq. (2.24) suggests, this leads to a decrease of the Jeans mass and thus to an even faster collapse. In particular, sub-sections of the cloud suddenly surpass their own Jeans limit and start collapsing on their own.

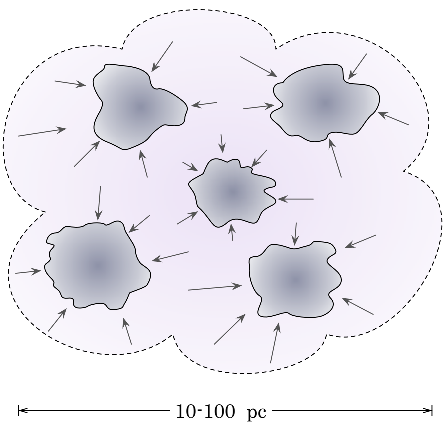

Figure 3.1: Fragmentation during the collapse of a GMC.

[2] Note that at one point we assumed ρ to be approximately constant and $p \sim \rho T$. On the other hand, equilibrium can in fact only be accomplished in the presence of a pressure gradient, thus $\rho \neq$ const. This should serve as an indication, that the above results only represent rough estimations.

This so called *fragmentation* process may go on and on until the cloud density becomes so high, that radiation can no longer easily escape and temperature starts to rise. The whole process becomes adiabatic and temperature behaves as $Tp^{-\frac{2}{5}}$: const, that is, $T \cdot (\rho k/\mu)^{-\frac{2}{3}}$: const and consequently

$$M_J \sim \frac{T^{\frac{3}{2}}}{\rho^{\frac{1}{2}}} \sim \sqrt{\rho} \quad . \tag{3.1}$$

This suggests that, as density increases and the process becomes more and more adiabatic, the Jeans mass increases and fragmentation eventually halts. The collapse slows down and may eventually result in the formation of stars or planets.

3.2 Estimating the maximum fragmentation size

A rough estimation for the minimum fragment size, can be given if one recalls that, for a given temperature T, the maximum energy Q radiated from a surface A is that of a grey body, namely

$$\frac{dQ_{\text{max}}}{dt} = \varepsilon A \sigma T^4 \quad , \tag{3.2}$$

with $\varepsilon \lesssim 1$ being the emissivity of the cloud. The times scales of the early collapse of a fragment, are comparable to its free fall time scale, given by

$$\tau_{\text{free}} = \sqrt{\frac{3\pi}{32G\rho}} \quad . \tag{3.3}$$

It accords approximately to the time needed for the fragment to contract to half its radius, at which point potential energy of the order $\Delta E \overset{(2.22)}{\approx} GM^2/R$ would have been released as heat and radiation. In order for this process to be isothermal, a radiation rate of the order

$$\frac{dQ_{\text{free}}}{dt} \approx \frac{\sqrt{8}}{\pi} \cdot \frac{G^{\frac{3}{2}} M^{\frac{5}{2}}}{R^{\frac{5}{2}}} \tag{3.4}$$

is needed. The process becomes adiabatic, and thus fragmentation halts, when $\dot{Q}_{\text{free}}$ approaches the upper limit in eq. (3.2), that is, when

$$M \approx \left(\sqrt{2}\varepsilon\sigma\pi^2\right)^{\frac{2}{5}} \cdot \frac{R^{\frac{9}{5}} T^{\frac{8}{5}}}{G^{\frac{3}{5}}} \quad . \tag{3.5}$$

Assuming that fragmentation stops, as soon as the Jeans mass M_J becomes equal to this limit mass, we obtain from eq. (2.24):

$$M_J^{\text{min}} \approx \frac{5^{\frac{9}{4}}}{\pi\sqrt{\sqrt{2}\varepsilon\sigma}} \cdot \frac{T^{\frac{1}{4}} k^{\frac{9}{4}}}{\mu^{\frac{9}{4}} G^{\frac{3}{2}}} \approx \frac{10}{\sqrt{\varepsilon\sigma}} \cdot \frac{T^{\frac{1}{4}} k^{\frac{9}{4}}}{\mu^{\frac{9}{4}} G^{\frac{3}{2}}} \approx \frac{0.025}{\sqrt{\varepsilon}} \cdot M_\odot \cdot \frac{T^{\frac{1}{4}}}{(\mu/u)^{\frac{9}{4}}} \ \text{K}^{-1} \quad . \tag{3.6}$$

where $u \approx 1.66 \times 10^{-27}$ Kg is the atomic mass unit. Assuming $\mu = 1$ u (Hydrogen), the temperature of the smallest fragment to be 10^3 K and the clouds emissivity $\varepsilon \approx 0.1$, we obtain a limit-mass of about $M_\odot/2$. This somewhat explains why fragmentation of GMCs typically terminates at the order of solar masses. For a more thorough treatment of the subject, we refer the reader to Rees[5].

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